FACULTY OF SCIENCES

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POINTWISE ANALYSIS OF RIEMANN'S OTHER FUNCTION

Frederik Broucke — fabrouck.broucke@ugent.be

based on joint work with J. Vindas

GHENT UNIVERSITY

Introduction

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}.$$

- At which points (if any) is f differentiable?
- Measure pointwise regularity?

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- 1996, Jaffard: lower bound $\alpha(x)$ at irrationals.

RELATION WITH THETA FUNCTION

Write $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$, and let

$$\phi(z) = \sum_{n=1}^{\infty} \frac{e(n^2 z)}{2\pi \mathrm{i} n^2}, \quad \theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z).$$

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$\phi'(z) = \frac{1}{2}(\theta(z) - 1)$, so

$$\phi(x+h) - \phi(x) + \frac{h}{2} = \frac{1}{2} \lim_{y \to 0^+} \int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta(x+z)\, \mathrm{d}z.$$

BEHAVIOR θ NEAR RATIONALS

Let $1 \leq p \leq q$, $(p,q) = 1$.

$$\theta\left(\frac{p}{q} + z\right) = \sum_{n\in\mathbb{Z}} e\left(\frac{pn^2}{q}\right) e(n^2 z) = \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \sum_{n\in j+q\mathbb{Z}} e(n^2 z)$$

$$= \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{q\sqrt{2}} z^{-1/2} \sum_{m\in\mathbb{Z}} S(q,p,m) \exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right),$$

with

$$S(q,p,m) = \sum_{j=1}^{q} e\left(\frac{pj^2 + mj}{q}\right).$$

Expansion ϕ at rationals

Using relation ϕ and θ, integrating by parts, and letting $y \to 0^+$, we obtain:

Theorem

For p and q integers, $q \geq 1$, $(p, q) = 1$

$$\phi\left(\frac{p}{q} + h\right) = \phi\left(\frac{p}{q}\right) + \frac{e^{i\pi/4}}{q\sqrt{2}} S(q, p) h^{1/2} - \frac{h}{2} + R_{q,p}(h),$$

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From the evaluation of $S(q, p)$ we get:

Corollary

ϕ is differentiable at p/q if and only if $q \equiv 2 \bmod 4$.

Hölder exponent at irrationals

Let ρ be irrational. Hölder exponent depends on Diophantine properties of ρ.
Let $r_n = p_n/q_n$ be the n-th convergent in continued fraction expansion of ρ.
Define τ_n via

$$|\rho - r_n| = \left(\frac{1}{q_n}\right)^{\tau_n}.$$

Let $(r_{n_k})_k$ be subsequence with $q_{n_k} \not\equiv 2 \mod 4$. Set

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Theorem

For irrational ρ, Hölder exponent is given by

$$\alpha(\rho) = \frac{1}{2} + \frac{1}{2\tau(\rho)}.$$

Upper bound

The upper bound

$$\alpha(\rho) \leq \frac{1}{2} + \frac{1}{2\tau(\rho)}$$

is due to Duistermaat.

Idea: consider subsequences of $r_{n_k} \to \rho$ and exploit square root behavior of ϕ at r_{n_k}.

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Using expansion of θ at r_n and properties of continued fractions, obtain

$$\theta(\rho + z) \ll_\varepsilon |z|^{\frac{1}{2\tau(\rho)} - \varepsilon - \frac{1}{2}} + y^{-1/2} |z|^{\frac{1}{2\tau(\rho)} - \varepsilon}.$$

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By Cauchy's formula, we have

$$\phi(\rho + h) - \phi(\rho) = -\frac{1}{2}h + \frac{1}{2}\int_\Gamma \theta(\rho + z)\,\mathrm{d}z,$$

Γ is boundary of rectangle with vertices h, $h + \mathrm{i}|h|$, $\mathrm{i}|h|$, and 0. Estimate the integral with the above bounds.

QUESTIONS?

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